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Free vibrations of a drop in partial contact with a solid support

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Under the assumptions of zero gravity, of negligible viscous effects and of small surface deformations, the problem of the axisymmetric free vibrations of a liquid drop immersed in an outer fluid and in partial contact with a spherical bowl has been analysed. Using the Green function method and expanding the velocity potentials in series of Legendre polynomials, the problem has been reduced to the solution of a single integral equation whose kernel has been expressed in analytical form. It is found that, in comparison with an isolated drop, constrained drops have an additional vibration mode which reduces to a zero-frequency rigid motion as the support size tends to zero, while the remaining ones approach the modes predicted by Lamb for a free drop.

The vibration modes have been numerically calculated for different sizes of the supported surface and compared with the experimental results of Bisch, Lasek & Rodot (1982).

1. Introduction

One of the fascinating prospects offered by future space stations is the possibility of obtaining monocrystals of high pureness and homogeneity exploiting the microgravity environment. Crystal-growth experiments have already been performed during Skylab and Salyut missions, and are planned for future Spacelab flights.

In order to grow pure crystals of high quality, it is important to understand the influence of vibrations on the growing crystal.

This problem may be reduced to the vibration analysis of a liquid drop in partial contact with a solid support representing the crystallized substratum.

Free vibrations of liquid drops have been extensively investigated in the case of isolated spheres. The first basic results date back to Kelvin (1890) and Rayleigh (1894) for the inviscid case. Later, in the case of a liquid sphere of density ρ^{i} surrounded by an outer fluid of density ρ^{e}, both fluids being inviscid or slightly viscous, Lamb (1932) obtained the following expression for the frequencies of small-amplitude vibrations:

$$\omega_n^2 = \frac{n(n-1)(n+1)(n+2)}{(n+1)\rho^{\mathrm{i}}+n\rho^{\mathrm{e}}}\frac{\sigma}{R^3}, \tag{1}$$

R being the drop radius and σ the surface tension between the two fluids. More recently the vibrations of isolated drops have been investigated in the full viscous case by Miller & Scriven (1968) and by Prosperetti (1980), while Foote (1971) extended the analysis to the case of large-amplitude vibrations.

However, the behaviour of a liquid drop in partial contact with a solid surface has not been studied extensively. For the sake of brevity, in the following discussion, we will refer to such a configuration as 'a constrained-drop problem'.

To our knowledge the only extensive investigation in this area has been performed by Bisch, Lasek and Rodot (Rodot, Bisch & Lasek 1979; Bisch, Lasek & Rodot 1982), who experimentally observed the axial vibrations of liquid drops lying on a cylindrical support with radius r_0 which could be varied independently of the drop radius. To simulate microgravity conditions they immersed the drop in an outer fluid having the same density, following the method first proposed by Plateau (1873). Besides the observations on the damping constants and on the amplitudes of oscillations, the main conclusion of their analysis is that the first resonance frequency turns out to be proportional to R^{-2} instead of $R^{-\frac{3}{2}}$ as in the case of an isolated drop.

However, since one would expect the behaviour of a constrained drop to approach that of a free drop as the support radius tends to zero, a question arises as to the validity range of the experimental correlation. In fact since the range of experimentally tested configurations was $0.14 < r_0/R < 0.77$ and the drop instability prevented further reduction of r_0/R, the answer to the above question can only be given through a theoretical analysis. This is one of the aims of the present work.

As mentioned above, the vibration modes of constrained drops have not yet been analysed on a theoretical basis. We should, however, mention the works of Benjamin & Scott (1979) and Benjamin (1981), where a similar problem, namely the oscillations in free-surface flows with constrained edges, has been analysed from a mathematical point of view. Following the main lines of the theory presented in the above-mentioned papers, the existence of an infinite sequence of real eigenvalues for the problem of constrained drops could be demonstrated and an estimate of their values could be given. However, the simple geometry of the constrained-drop problem suggests a different approach, which leads to an explicit form of the operators and reduces the problem to the solution of a system of algebraic equations. This method allows for a simple and accurate evaluation of the constrained-drop eigenvalues and eigenmodes in order to study their dependence on the relevant parameters of the problem and to make a meaningful comparison with experimental data.

2. The mathematical model

2.1. *Mathematical formulation of the problem*

The free vibrations of a spherical drop, immersed in an immiscible fluid of different density and in partial contact with a solid support, are considered. A spherical polar coordinate system (r, θ, ψ) is adopted, and only symmetrical deformations of the drop with respect to the y-axis are considered. The resulting geometric configuration of the problem is indicated in figure 1. Gravity and viscous-dissipation effects are neglected. The inner and outer fluids are assumed to be incompressible, and only small vibrations of the free surface are considered, i.e. $|z/R| \ll 1$.

In what follows the superscripts i and e will indicate quantities pertaining to the internal and external fluid respectively. The absence of the superscript indicates that a statement is applicable to both fluids.

The velocity field is described by the potential ϕ, which satisfies the Laplace equation

$$\nabla^2\phi = \sin\theta\,(r^2\phi_r)_r + (\sin\theta\,\phi_\theta)_\theta = 0 \tag{2}$$

on

$$\left.\begin{aligned} \mathscr{D}^{\mathrm{i}} &\equiv \{(r,\theta)|\, 0 \leqslant r \leqslant R,\, 0 \leqslant \theta \leqslant \pi\}, \\ \mathscr{D}^{\mathrm{e}} &\equiv \{(r,\theta)|\, R \leqslant r \leqslant +\infty,\, 0 \leqslant \theta \leqslant \pi\}, \end{aligned}\right\} \tag{3}$$

where account has been taken of the assumed symmetry, and the definition of the domains $\mathscr{D}$ follows from the hypothesis of small free-surface deformations about the

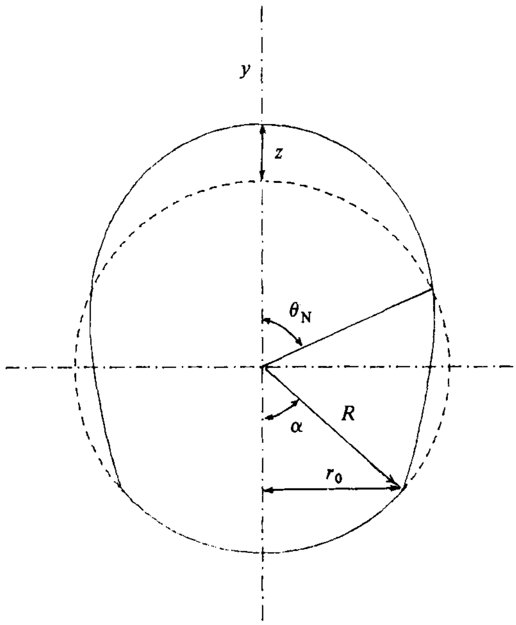

FIGURE 1. Definition sketch.

spherical shape. The pressure $p(r, \theta, t)$ may be obtained by the linearized Bernoulli equation

$$p = p_0 + \rho\phi_t. \tag{4}$$

On the boundary portion $\partial\mathscr{D}^s \equiv \{(r,\theta)\,|\,r = R,\ \pi-\alpha \leqslant \theta \leqslant \pi\}$ where the drop is in contact with the support, we assume the normal velocity and the free-surface deformation $Z(\theta, t)$ to be zero, that is

$$\phi_r = 0, \quad Z = 0. \qquad (5), (6)$$

Moreover, on the free portion $\partial\mathscr{D}^f \equiv \{(r,\theta)\,|\,r = R,\ 0 \leqslant \theta \leqslant \pi-\alpha\}$ of the boundary, we shall impose the continuity of the normal velocity across the free surface,

$$\phi_r = -Z_t, \tag{7}$$

and the dynamic balance of momentum along the drop radius,

$$p^i - p^e = \sigma\left[\frac{2}{R} - \frac{1}{R^2}\left(\frac{1}{\sin\theta}(\sin\theta\, Z_\theta)_\theta + 2Z\right)\right], \tag{8}$$

where the quantity between square brackets is the local curvature of the deformed free surface. An integral condition on the free-surface shape is moreover given by the conservation of the drop mass, that is

$$\int_0^\pi Z\sin\theta\,\mathrm{d}\theta = 0. \tag{9}$$

By considering periodic motion we can set

$$\phi = \varphi(r,\theta)\,\mathrm{e}^{\mathrm{i}\omega t}, \quad Z = z(\theta)\,\mathrm{e}^{\mathrm{i}(\omega t + \frac{1}{2}\pi)}, \qquad (10a, b)$$

where the $\frac{1}{2}\pi$ phase angle between ϕ and Z is consistent with the boundary condition (7).

By substituting (10) into (8) and by equating the real parts, one easily finds

$$p_0^{\mathrm{i}} - p_0^{\mathrm{e}} = \frac{2\sigma}{R}; \tag{11}$$

that is, the well-known Laplace formula for the unperturbed spherical drop.

The system (2)–(9), on account of (10) and (11), gives

$$\nabla^2 \varphi = 0 \quad \text{on} \quad \mathscr{D}, \tag{12}$$

$$\varphi_r = \omega z, \quad \omega(\rho^{\mathrm{i}}\varphi^{\mathrm{i}} - \rho^{\mathrm{e}}\varphi^{\mathrm{e}}) = -\frac{\sigma}{R^2}\left[\frac{1}{\sin\theta}(\sin\theta\, z_\theta)_\theta + 2z\right] \quad \text{on} \quad \partial\mathscr{D}^{\mathrm{f}}, \tag{13a, b}$$

$$\varphi_r = 0, \; z = 0 \quad \text{on} \quad \partial\mathscr{D}^{\mathrm{s}}, \tag{14a, b}$$

$$\int_0^\pi z \sin\theta \, \mathrm{d}\theta = 0. \tag{15}$$

Equations (12)–(15) determine an eigenvalue problem, which is reduced in §§2.2–2.6, to a more compact form, which enables the use of a simple approximate numerical method of solution.

2.2. *The solutions for inner and outer potentials*

It is convenient to introduce the variable $\mu = \cos\theta$. If $z(\mu)$ is a given function in the space $L^2(-1, 1)$ of square-integrable functions which satisfies the conditions (14*b*) and (15), the problem (12), (13*a*), (14*a*) is easily recognized to be equivalent to the Neumann problem

$$\nabla^2\varphi = 0 \quad \text{on} \quad \mathscr{D},$$

$$\varphi_r = \omega z \quad \text{on} \quad \partial\mathscr{D},$$

$$\varphi^{\mathrm{e}} \to 0 \quad \text{as} \quad r \to \infty,$$

whose solution is shown, by standard methods, to be

$$\varphi^{\mathrm{i}}(r, \mu) = \omega R\left(\varphi_0 + \sum_{k=1}^{\infty} \frac{\varphi_k}{k}\frac{r^k}{R^k} P_k(\mu)\right), \tag{16a}$$

$$\varphi^{\mathrm{e}}(r, \mu) = -\omega R \sum_{k=1}^{\infty} \frac{\varphi_k}{k+1}\frac{R^{k+1}}{r^{k+1}} P_k(\mu). \tag{16b}$$

Here $P_k(\mu)$ is the kth Legendre polynomial, φ_0 a constant to be determined in the following, and the coefficients φ_k are given by

$$\varphi_k = \frac{\langle z, P_k\rangle}{\langle P_k, P_k\rangle}, \tag{16c}$$

where

$$\langle f, g\rangle = \int_{-1}^{1} fg \, \mathrm{d}\mu$$

denotes the inner product in $L^2(-1, 1)$, and $\langle P_k, P_k\rangle = 2/(2k+1)$.

2.3. *Integral equation for the surface deformation*

The unknown free-surface shape satisfies the differential equation (13*b*) which under the change of variable $\mu = \cos\theta$ becomes

$$[(1-\mu^2)\, z\mu]_\mu + 2z = -\frac{\rho^{\mathrm{i}}\omega R^2}{\sigma}\left[\varphi^{\mathrm{i}}(R, \mu) - \frac{\rho^{\mathrm{e}}}{\rho^{\mathrm{i}}}\varphi^{\mathrm{e}}(R, \mu)\right] = f(\mu) \quad \text{for} \quad a \leqslant \mu \leqslant 1, \tag{17a}$$

with the boundary condition

$$z(a) = 0. \tag{17b}$$

The solution of (17) may be easily found by the use of the Green-function method. In fact, when $f(\mu) = 0$ (17a) has the following two independent solutions:

$$P_1(\mu) = \mu, \quad Q_1(\mu) = \tfrac{1}{2}\mu \ln\frac{1+\mu}{1-\mu} - 1. \tag{18a, b}$$

The solution of (17) may thus be expressed as

$$z(\mu) = \int_a^1 G'(\mu,\tau) f(\tau)\,\mathrm{d}\tau \quad \text{for} \quad a \leqslant \mu \leqslant 1, \tag{19a}$$

where the symmetric Green function $G'(\mu,\tau)$: $[a,1]\times[a,1] \rightarrow \mathbb{R}$ is given by

$$G'(\mu,\tau) = \left\{ \begin{array}{ll} P_1(\tau)\left[\frac{Q_1(a)}{a}P_1(\mu) - \frac{P_1(a)}{a}Q_1(\mu)\right] & \text{for} \quad a \leqslant \mu \leqslant \tau \leqslant 1, \\ P_1(\mu)\left[\frac{Q_1(a)}{a}P_1(\tau) - \frac{P_1(a)}{a}Q_1(\tau)\right] & \text{for} \quad a \leqslant \tau \leqslant \mu \leqslant 1. \end{array} \right\} \tag{19b}$$

When $a \rightarrow 0$ or $a \rightarrow -1$ the Green function diverges. These two cases will be discussed later.

The function $z(\mu)$, as given by (19a), is such that

$$\lim_{\mu \rightarrow a^+} z(\mu) = 0.$$

When $-1 \leqslant \mu \leqslant a$ the boundary condition (14b) gives $z(\mu) = 0$. Equation (19a) may then be replaced by

$$z(\mu) = \int_{-1}^1 G(\mu,\tau) f(\tau)\,\mathrm{d}\tau, \tag{20a}$$

where $G(\mu,\tau)$: $[-1,1]\times[-1,1] \rightarrow \mathbb{R}$ is the extension of $G'(\mu,\tau)$ defined by

$$\left. \begin{array}{l} G(\mu,\tau) = G'(\mu,\tau) \quad \text{for} \quad (\mu,\tau) \in [a,1]\times[a,1], \\ G(\mu,\tau) = 0 \quad \text{otherwise.} \end{array} \right\} \tag{20b}$$

2.4. *The eigenvalue problem*

On account of (16) we may evaluate the function $f(\tau)$ appearing in (20a) as

$$f(\mu) = -\frac{1}{\lambda}\left[\varphi_0 + \int_{-1}^1 \Gamma(\mu,\tau)\, z(\tau)\,\mathrm{d}\tau\right], \tag{21a}$$

where

$$\lambda = \frac{\sigma}{\rho^{\mathrm{i}}\omega^2 R^3} \tag{21b}$$

and

$$\Gamma(\mu,\tau) = \sum_{k=1}^{\infty} \frac{2k+1}{2}\left[\frac{1}{k} + \frac{\rho^{\mathrm{e}}}{\rho^{\mathrm{i}}}\frac{1}{k+1}\right] P_k(\mu)\, P_k(\tau). \tag{21c}$$

Equation (20a), after substitution of (21a), becomes

$$z(\mu) = -\frac{1}{\lambda}\left[\varphi_0 \int_{-1}^1 G(\mu,\tau)\,\mathrm{d}\tau + \int_{-1}^1\int_{-1}^1 G(\mu,\tau)\,\Gamma(\tau,\sigma)\, z(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right]. \tag{22}$$

The constant φ_0 is now determined by imposing the condition on the conservation of the total drop volume. By substituting (22) into (15), in fact, we obtain

$$\varphi_0 = \frac{-\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} G(\nu,\tau)\,\Gamma(\tau,\sigma)\,z(\sigma)\,\mathrm{d}\nu\,\mathrm{d}\sigma\,\mathrm{d}\tau}{\int_{-1}^{1}\int_{-1}^{1} G(\nu,\tau)\,\mathrm{d}\nu\,\mathrm{d}\tau}, \tag{23}$$

and (22) is then reduced to the form

$$\lambda z(\mu) = \int_{-1}^{1} K(\mu,\sigma)\,z(\sigma)\,\mathrm{d}\sigma = \mathscr{A}z \quad (-1 \leqslant \mu \leqslant 1), \tag{24a}$$

where $K(\mu,\sigma)$ is given by

$$K(\mu,\sigma) = \frac{\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} G(\mu,\epsilon)\,G(\nu,\tau)\,\Gamma(\tau,\sigma)\,\mathrm{d}\tau\,\mathrm{d}\nu\,\mathrm{d}\epsilon}{\int_{-1}^{1}\int_{-1}^{1} G(\nu,\tau)\,\mathrm{d}\tau\,\mathrm{d}\nu} - \int_{-1}^{1} G(\mu,\tau)\,\Gamma(\tau,\sigma)\,\mathrm{d}\tau. \tag{24b}$$

In order to reduce (24) to an infinite system of linear algebraic equations we expand $z(\mu)$ and $K(\mu,\sigma)$ in the Legendre-polynomial series

$$z(\mu) = \sum_{k=1}^{\infty} z_k P_k(\mu), \quad z_k = \frac{\langle z, P_k\rangle}{\langle P_k, P_k\rangle} \tag{25a}$$

and

$$K(\mu,\sigma) = \sum_{h,\,l=1}^{\infty} K_{hl}\,P_h(\mu)\,P_l(\sigma), \tag{25b}$$

where the $P_0(\mu)$ term has been dropped as a consequence of (15). The coefficients K_{hl} may be evaluated from the corresponding ones of the expansion

$$G(\mu,\sigma) = \sum_{h,\,l=0}^{\infty} G_{hl}\,P_h(\mu)\,P_l(\sigma), \tag{26a}$$

where, on account of the definition (20*b*),

$$G_{hl} = G_{lh} = \frac{\langle\langle G, P_h\rangle, P_l\rangle}{\langle P_h, P_h\rangle\langle P_l, P_l\rangle} = \frac{(2h+1)(2l+1)}{4}\int_a^1\int_a^1 G'(\mu,\sigma)\,P_h(\mu)\,P_l(\sigma)\,\mathrm{d}\mu\,\mathrm{d}\sigma. \tag{26b}$$

In fact, the substitution of (26) and of the expansion (21*c*) for $\Gamma(\tau,\sigma)$ into (24*b*) results in

$$K_{hl} = \left(\frac{G_{0l}G_{h0}}{G_{00}} - G_{lh}\right)\left(\frac{1}{l} + \frac{\rho^{\mathrm{e}}}{\rho^{\mathrm{i}}}\frac{1}{l+1}\right). \tag{27}$$

On account of (25), (24*a*) finally gives

$$\lambda z_h = \sum_{l=1}^{\infty}\left[\frac{2}{2l+1}\left(\frac{G_{h0}G_{0l}}{G_{00}} - G_{hl}\right)\left(\frac{1}{l} + \frac{1}{l+1}\right)\right] z_l, \tag{28}$$

which, by setting

$$x_h = \left(\frac{\frac{1}{h} + \frac{\rho^{\mathrm{e}}}{\rho^{\mathrm{i}}}\frac{1}{h+1}}{2h+1}\right)^{\frac{1}{2}} z_h, \tag{29a}$$

is equivalent to

$$\lambda x_h = \sum_{l=1}^{\infty} A_{lh}\,x_l, \tag{29b}$$

where

$$A_{lh} = A_{hl} = 2\left(\frac{G_{h0}\,G_{0l}}{G_{00}} - G_{hl}\right)\left[\frac{\left(\frac{1}{h}+\frac{\rho^{e}}{\rho^{i}}\frac{1}{h+1}\right)}{2h+1}\;\frac{\left(\frac{1}{l}+\frac{\rho^{e}}{\rho^{i}}\frac{1}{l+1}\right)}{2l+1}\right]. \tag{29c}$$

2.5. *Solution procedure*

The coefficients G_{ik} $(i,k = 0,1,2,\ldots)$, and hence, by the use of (29*c*), the coefficients A_{lh} $(l,h = 1,2,\ldots)$, have been calculated on the basis of well-known properties of the Legendre functions of first $(P_k(\mu))$ and second $(Q_k(\mu))$ kind.

In fact, by substituting (19*b*) into (26*b*) we obtain

$$G_{ik} = \frac{(2i+1)(2k+1)}{4}\left\{\left[\int_a^1 P_1(\tau)\,P_i(\tau)\,\mathrm{d}\tau\right]\left[\int_a^1\left(\frac{Q_1(a)}{a}P_1(\mu)-\frac{P_1(a)}{a}Q_1(\mu)\right)P_k(\mu)\,\mathrm{d}\mu\right]\right.$$
$$\left.+\frac{P_1(a)}{a}\int_a^1\left[P_i(\tau)\,P_1(\tau)\int_\tau^1 Q_1(\mu)\,P_k(\mu)\,\mathrm{d}\mu - Q_1(\tau)\,P_i(\tau)\int_\tau^1 P_1(\mu)\,P_k(\mu)\,\mathrm{d}\mu\right]\mathrm{d}\tau\right\}. \tag{30}$$

When $K \neq 1$ the expression (30) may be simplified by the use of the identities (MacRobert 1967)

$$\int_x^1 P_h(\mu)\,P_k(\mu)\,\mathrm{d}\mu = \frac{hP_k(x)\,P_{h-1}(x)-kP_h(x)\,P_{k-1}(x)-(h-k)\,xP_h(x)\,P_x(x)}{(h-k)(h+k+1)}, \tag{31a}$$

$$\int_x^1 Q_h(\mu)\,P_k(\mu)\,\mathrm{d}\mu = \frac{hP_k(x)\,Q_{h-1}(x)-kQ_h(x)\,Q_{k-1}(x)-(h-k)\,xQ_h(x)\,P_k(x)}{(h-k)(h+k+1)}. \tag{31b}$$

After some straightforward algebra one then obtains, when $k \neq 1$,

$$G_{ki} = G_{ik} = \frac{(2i+1)(2k+1)}{4}\,\frac{P_k(a)\int_a^1 P_1(\tau)\,P_i(\tau)\,\mathrm{d}\tau - P_1(a)\int_a^1 P_i(\tau)\,P_k(\tau)\,\mathrm{d}\tau}{a[k(k+1)-2]}, \tag{32a}$$

which may be easily evaluated analytically; while, when $i = k = 1$, (30) gives

$$G_{11} = \tfrac{1}{2}\ln(1+a) - \frac{1}{4a} + \frac{7-6\ln 2}{12} - \frac{a^3}{12} - \frac{a}{4}. \tag{32a}$$

2.6. *The eigenvalue problem for $a = -1$ and $a = 0$*

Where $a = -1$ or $a = 0$ the Green function (19*b*) diverges. Therefore the analysis of the eigenvalue problem (12)–(15) may no longer be accomplished in the function space $L^2(-1,1)$, but must be done in different function spaces (Benjamin 1981).

A mathematical analysis of these cases is outside the purposes of the present work. It is nevertheless possible to analyse the behaviour of the operator $\mathscr{A}$ and of its eigenvalues as $a \to 0$ and $a \to -1^+$.

The first case is relatively simple. In fact, as shown in the Appendix, the $\lim_{a\to 0} A_{hk}$ exists and is finite. The singularity for $a = 0$ is thus easily dropped by setting

$$\mathscr{A}_{a=0} = \lim_{a\to 0}\mathscr{A}_a.$$

The same cannot be done when $a = -1$. In fact, from (30) we get, by using well-known properties of the Legendre polynomials,

$$\lim_{a\to -1^+} G_{ik} = \frac{(2i+1)(2k+1)}{4} \frac{(-1)^k \frac{2\delta_{i1}}{3} + \frac{2\delta_{ik}}{2k+1}}{2-k(k+1)}, \tag{33a}$$

while G_{11} diverges as $\ln(1+a)$ when $a \to -1$.

The matrix A_{ik} has therefore, in the limit $a \to -1$, an 'arrow-shaped' structure, that is

$$A_{11} \to +\infty,$$

$$A_{1k} = A_{k1} \to \frac{(-1)^k}{k(k+1)-2}\left[\left(1+\frac{1}{2}\frac{\rho^e}{\rho^i}\right)\left(\frac{1}{k}+\frac{1}{k+1}\frac{\rho^e}{\rho^i}\right)\left(\frac{2k+1}{3}\right)\right]^{\frac{1}{2}} \quad (k \neq 1),$$

$$A_{hk} \to 0 \quad (h \neq k, h, \ k \neq 1),$$

$$A_{hh} \to \left(\frac{1}{h}+\frac{\rho^e}{\rho^i}\frac{1}{h+1}\right)\frac{1}{h(h+1)-2} \quad (h \neq 1).$$

The following behaviour of the eigenvalues and eigenvectors of the matrix A_{ik} can thus be seen:

$$\lambda^1 \to +\infty, \quad x_k^1 \to \delta_{1k}, \quad z_k^1 \to \delta_{1k}, \tag{34a}$$

$$\lambda^n \to \frac{(n+1)+(\rho^e/\rho^i)\,n}{n(n-1)(n+1)(n+2)}, \quad x_k^n \to \delta_{nk}, \quad z_k^n \to \delta_{nk}. \tag{34b}$$

The first eigenmode thus tends to the first Legendre polynomial with a frequency $\omega^1 \to 0$, that is to a rigid displacement of the drop, while the remaining eigenmodes and eigenvalues tend to the corresponding ones for the drop in absence of the solid support (Lamb 1932).

We can thus argue that the solid support, besides the modification of the frequencies and of the modes with respect to the case of the unrestrained drop, introduces a new low-frequency mode, which tends to a zero-frequency rigid displacement as the amplitude of the supported portion of the drop reduces to zero.

3. Results and discussion

The first four computed eigenmodes for three different values of the characteristic parameter $\beta = r_0/R$ are compared in figure 2 with the eigenmodes of the isolated drop and with the drop configurations experimentally observed by Bisch *et al.* (1982). The liquids used in this experiment were benzene and carbon tetrachloride in water, having a viscosity of 0.9 and 1 cSt respectively. The low values of the viscosity makes the comparison with the present inviscid results meaningful. In spite of the different geometry of the support (a cylinder in the experiments and a spherical bowl in the calculations) a close resemblance is observed between the experimental (figure 2*a*) and the computed (figure 2*b*) configurations having the same value of β. As the support radius tends to zero we observe a significant change of the calculated drop shape, which approaches (figure 2*d*) the configuration of a freely vibrating drop (figure 2*e*) (Lamb 1932), as was anticipated in §2.

It should be noticed that the first vibration mode for the free drop corresponds to the second one ($n = 2$) of the constrained drop. In fact, for $n = 1$ the first mode of the constrained drop reduces, as $\beta \to 0$, to a rigid zero-frequency displacement (see §2.6).

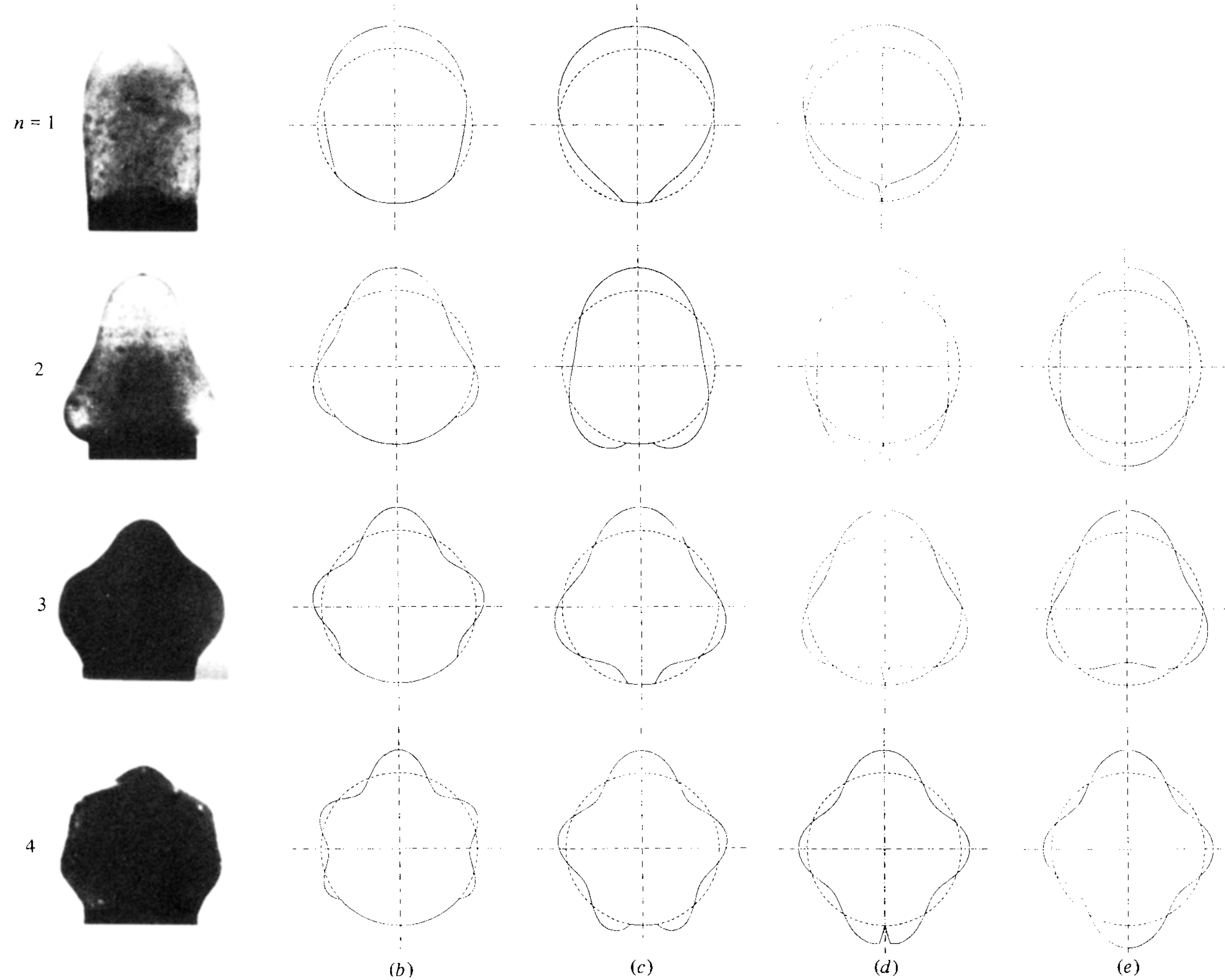

FIGURE 2. Vibration modes at the first four resonance frequencies: (*a*) experimental observations by Bisch *et al.* (1982) at $r_0/R = 0.75$; (*b*) computed configurations at $r_0/R = 0.75$; (*c*) computed configurations at $r_0/R = 0.17$; (*d*) computed configurations at $r_0/R = 10^{-4}$; (*e*) free-drop configurations.

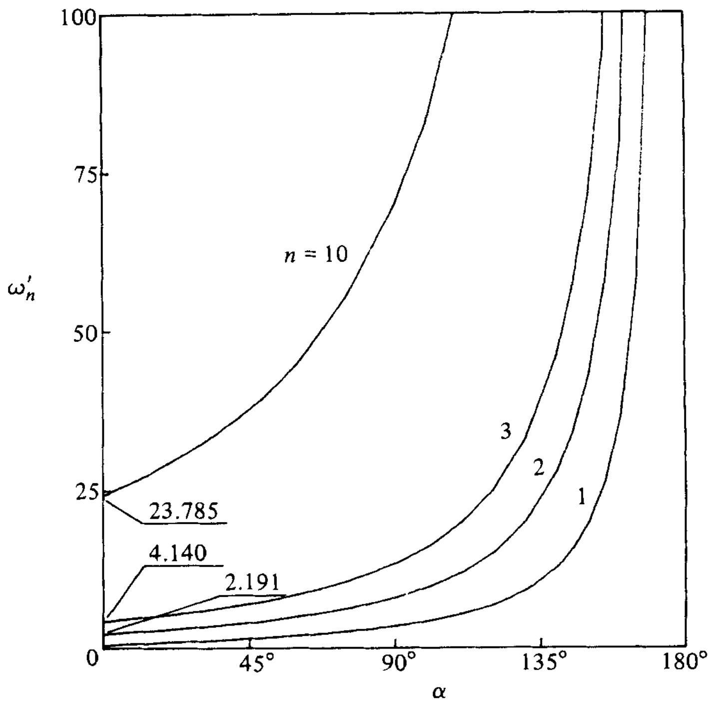

FIGURE 3. Non-dimensional frequency ω'_n versus the support angle α for $n = 1, 2, 3, 10$.

A second remark concerning the vibration modes is the surprising similarity that can occur between the shapes of drops having different wavenumber and different β-values, as can be observed in figure 2, comparing the configurations (*a*) for $n = i$ with the configurations (*e*) for $n = i+1$, $i = 1, 2, 3$.

This fortuitous and misleading similarity has probably led Bisch *et al.* (1982) to compare the first resonance frequency that they detected experimentally with the $n = 2$ Lamb's frequency, while the correct comparison must be done between vibration modes having the same wavenumber.

This remark is also confirmed by the results reported in figure 3, where the computed non-dimensional frequency

$$\omega'_n = \left(\frac{\rho^i R^3}{\sigma}\right)^{\frac{1}{2}} \omega_n = \lambda_n^{-\frac{1}{2}}$$

is plotted versus the angle $\alpha = \sin^{-1}\beta$.

It may be observed that when $\alpha \rightarrow \pi$ all frequencies tend to infinity. From a geometrical point of view, however, the case of $\alpha > \frac{1}{2}\pi$ may be more appropriately considered as a spherical meniscus rather than a constrained drop. On the other hand, when $\alpha \rightarrow 0$ all the constrained drop frequencies approach the values predicted by Lamb's theory.

We thus observe that the solid support increases the resonance frequency for a given wavenumber, but introduces an additional low-frequency mode. It follows that the first resonance frequency, for low values of the ratio of support to drop radius, may be even smaller than the first for the free drop, a fact that may be of practical importance in the applications.

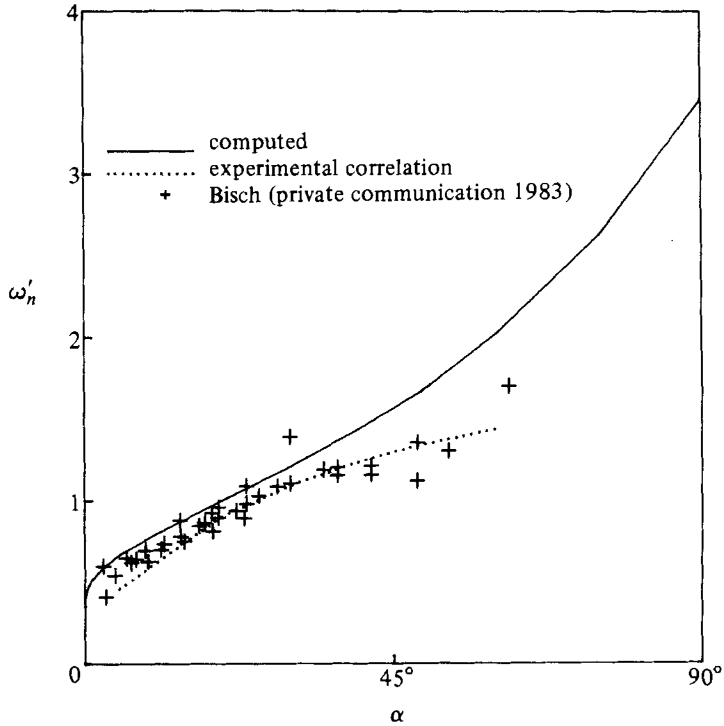

FIGURE 4. Comparison between experimental and present results for the first resonance frequency.

It may finally be pointed out that, even if not evident from the representation of figure 3, in the neighbourhood of $\alpha = 0$ all curves have a vertical slope. This behaviour can be better observed in figure 4, where the first vibration frequency is plotted on a different scale and compared with the experimental results of Bisch (private communication 1983) using two fluids with viscosities of 4.4 and 1 cSt respectively. For values of α up to 40° the agreement with the experimental results is quite satisfactory, particularly if we consider that the present theory is based on the inviscid assumption. In fact, as shown by Prosperetti (1980) for the free drop, the effect of the viscosity is to reduce slightly the values of the resonance frequencies. On the other hand, an increasing disagreement between the theoretical and experimental results occurs at larger values of the angle α, which seems to indicate the departure of the actual behaviour of the drop from the simple mathematical model here proposed. A deeper investigation and further experimental evidence in the range of large values of α is required to clarify this behaviour.

To analyse the frequency dependence on the drop radius for a given support dimension we introduce the non-dimensional frequency

$$\omega_n'' = \left(\frac{\rho^i r_0}{\sigma}\right)^{\frac{1}{2}} \omega_n = \left(\frac{\sin^3 \alpha}{\nu n}\right)^{\frac{1}{2}}.$$

The logarithmic plot of figure 5 indicates that a correlation form like

$$\omega_n'' = aR^{-\gamma}$$

may be assumed with a constant γ value only in a limited range of R-values. In fact for $R \rightarrow \infty$ (i.e. $\alpha \rightarrow 0$) the coefficient γ tends to the value $\frac{3}{2}$ predicted by Lamb's theory

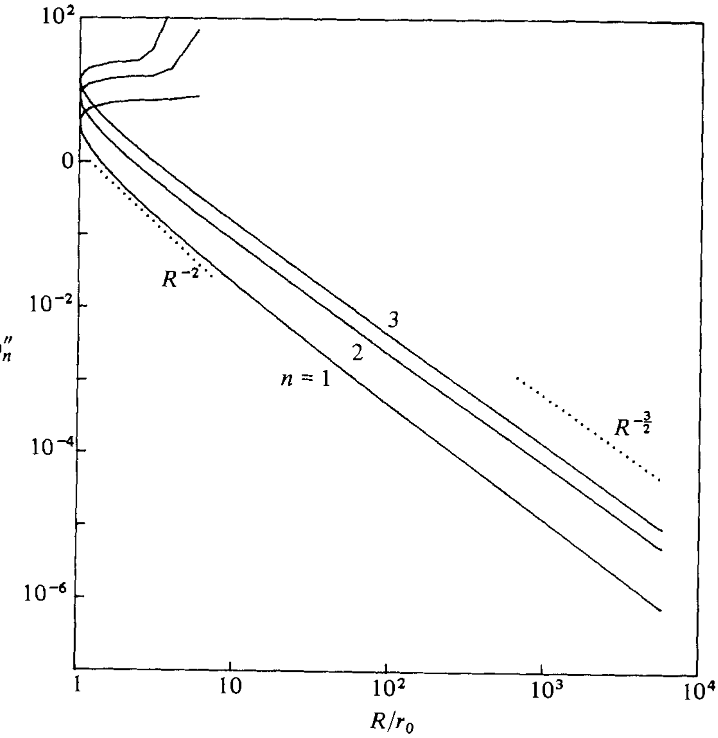

FIGURE 5. Logarithmic plot of the non-dimensional frequency ω_n'' versus R/r_0 at $n = 1, 2, 3$.

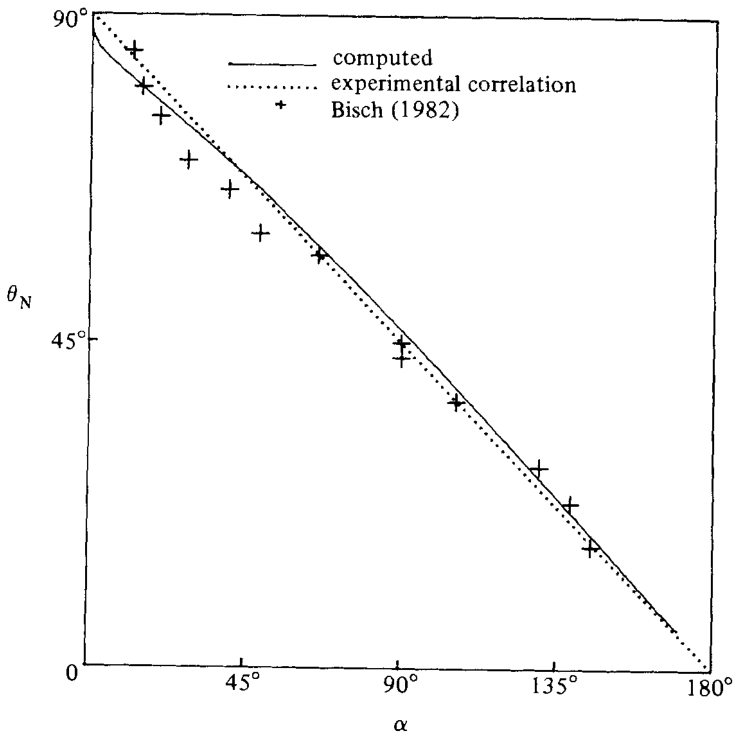

FIGURE 6. Nodal-circle position for $n = 1$ versus the support angle α.

for a free drop, while for $R = r_0$ the coefficient γ has an infinite value. In the range of R where experiments have been performed (i.e. $1.3r_0 < R < 7r_0$) the computed γ-value ranges between 1.75 and 2.9 consistently with the value of 2 proposed by Bisch *et al.* (1982).

A further comparison with the experimental results may be done considering the nodal circle position of the first eigenmode. Bisch (1982) has experimentally observed that the nodal circle position defined by the angle θ_N indicated in figure 1 is related to the angle α by the very simple relation

$$\theta_N = \tfrac{1}{2}(\pi - \alpha).$$

This correlation together with the experimental results is compared in figure 6 with our results. The agreement is shown to be satisfactory except for very small values of the α-angle, where the computed curve exhibits the same behaviour observed in figure 3 for the frequency.

On account of the mass conservation, this behaviour is connected to the fact that when $\alpha \approx 0$ the shape of the perturbed surface has not a linear variation with the support size, as it is shown by the shape of the small stalk in figure 2(*d*). Even for very small values of α the continuity of the function makes the eigenmode very different from the discontinuous solution that would occur at $\alpha = 0$. In the latter case in fact the first eigenmode would be zero along all the drop surface, with the exception of the contact point, where it would have a finite value, giving rise to a tadpole-shaped drop. Consequently the fulfilment of condition (15) requires a displacement of the nodal circle that can have finite values even for infinitesimal values of α. The same interpretation may be used to explain the vertical slope for $\alpha = 0$ of the eigenvalues curves of figures 3 and 4.

We finally discuss the influence on the resonance frequencies of the outer fluid density ρ^e.

If we indicate by ω_n^0 the frequencies for $\rho^e = 0$, the expression (1) for the free drops may be written in the form

$$\tau = \left(\frac{\omega_n^0}{\omega_n}\right)^2 = 1 + \gamma_n \frac{\rho^e}{\rho^i}.$$

The same quantity τ has been computed and plotted in figure 7 for different values of the wavenumber and of the angle α. It is shown that in all conditions τ varies linearly with the density ratio and that the slope γ_n increases both with n and α. It may therefore be concluded that, for a constrained drop, an increase of the outer fluid density gives rise to a frequency reduction larger than that for a free drop.

The authors wish to thank Dr Christian Bisch, who proposed the problem and has kindly supplied the experimental results.

Appendix

In this appendix, the limit as $a \to 0$ of the coefficients A_{hk} in (29*c*) is shown to exist and to be finite.

We have

$$\lim_{a \to 0} A_{hk} = 2\left(\frac{\dfrac{1}{h} + \dfrac{\rho^e}{\rho^i}\dfrac{1}{h+1}}{2h+1}\,\frac{\dfrac{1}{k} + \dfrac{\rho^e}{\rho^i}\dfrac{1}{k+1}}{2k+1}\right)^{\frac{1}{2}} \lim_{a \to 0}\left(\frac{G_{h0}G_{k0}}{G_{00}} - G_{hk}\right) \tag{A 1}$$

and

$$\lim_{a \to 0}\left(\frac{G_{h0}G_{k0}}{G_{00}} - G_{hk}\right) = \lim_{a \to 0}\left[\frac{1}{a}\left(\frac{aG_{h0}aG_{k0}}{aG_{00}} - aG_{hk}\right)\right]. \tag{A 2}$$

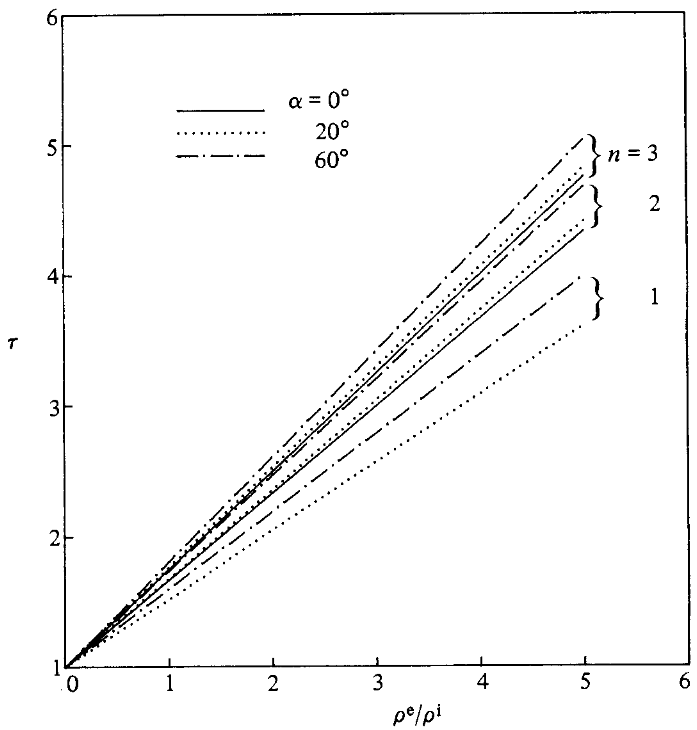

FIGURE 7. Influence of the outer to inner density ratio on the function $\tau = (\omega_n/\omega_n^0)^{-2}$.

An expansion in a Taylor series of the quantities aG_{hk} near $a = 0$ results in

$$aG_{hk} = \frac{(2h+1)(2k+1)}{4}\left[\frac{P_k(0)\int_0^1 P_1(\tau)\,P_h(\tau)\,d\tau}{k(k+1)-2}\right.$$

$$\left.+\frac{P_k(0)\int_0^1 P_1(\tau)\,P_h(\tau)\,d\tau-\int_0^1 P_h(\tau)\,P_k(\tau)\,d\tau}{k(k+1)-2}\,a+O(a^2)\right] \quad \text{(A 3}a\text{)}$$

when $k \neq 1$, and in

$$aG_{11} = \tfrac{9}{4}[-\tfrac{1}{9}+\tfrac{1}{27}(7-6\ln 2)\,a+O(a^2)]. \quad \text{(A 3}b\text{)}$$

It follows that, for each h and k,

$$\left(\frac{aG_{h0}\,aG_{k0}}{aG_{00}}-aG_{hk}\right)_{a=0} = 0,$$

and that the derivative of the above quantity at $a = 0$ is finite, thus showing the existence of a finite value for the limit in (A 2).

For completeness we list here the explicit expression that is found for these limits:

$$\lim_{a\to 0}\left(\frac{G_{h0}\,G_{0k}}{G_{00}}-G_{hk}\right) = \frac{(2h+1)(2k+1)}{4[k(k+1)-2]}\left[P_k(0)\left(2\int_0^1 P_1(\tau)\,P_h(\tau)\,d\tau-\int_0^1 P_h(\tau)\,d\tau\right)\right.$$

$$\left.-2\int_0^1 P_1(\tau)\,P_h(\tau)\,d\tau\int_0^1 P_k(\tau)\,d\tau+\int_0^1 P_h(\tau)\,P_k(\tau)\,d\tau\right] \quad \text{(A 4}a\text{)}$$

when $k \neq 1$, and

$$\lim_{a \to 0}\left(\frac{G_{10}\,G_{10}}{G_{00}} - G_{11}\right) = -\tfrac{1}{3} + \tfrac{1}{2}\ln 2. \qquad \text{(A 4}b\text{)}$$

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